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QUANTUM DYNAMICS FOR DE SITTER RADIATION

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We revisit the Hamiltonian formalism for a massive scalar field and study the particle production in a de Sitter space. In the invariant-operator picture the time-dependent annihilation and creation operators are constructed in terms of a complex solution to the classical equation of motion for the field and the Gaussian wave function for each Fourier mode is found which is an exact solution to the Schrödinger equation. The in-out formalism is reformulated by the annihilation and creation operators and the Gaussian wave functions. The de Sitter radiation from the in-out formalism differs from the Gibbons-Hawking radiation in the planar coordinates, and we discuss the discrepancy of the particle production by the two methods.

Keywords: de Sitter Space; Particle Production; Gaussian Wave Packets; In-Out Formalism

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1. Introduction

The universe with a cosmological constant is the pure de Sitter (dS) space and has the maximal symmetry that makes a quantum field theory more tractable than other curved spacetimes (for a review and references, see ref. [1]). In spite of numerous works since the discovery of the dS-invariant vacuum by Bunch and Davies² and the dS radiation by Gibbons and Hawking³ (for instance, see ref. [4]), the proper selection of the physical vacuum of a quantum field in dS space has recently been challenged by Polyakov.⁵ It has been further argued that the cosmological constant problem may be resolved by IR physics through the UV/IR mixing from an interaction.⁶

In this paper, we revisit the quantum field theory for a massive scalar field in the dS space by unifying the in-out formalism by Schwinger and DeWitt^{7,8} and the invariant-operator picture by Lewis and Riesenfeld.⁹ The in-out formalism provides a good framework for quantum field theory involving particle production. It has been known that an expanding spacetime produces particles¹⁰ and that the dS space has the Gibbons-Hawking radiation^{3,4} (for a recent discussion, see ref. [11]).

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Recently the dS radiation has been interpreted in connection with the Schwinger mechanism.^{5,6,12,13} In contrast to the early works on the in-out formalism,¹⁴ in this paper we shall directly use the wave functions for the field and shall find the vacuum persistence amplitude between two spacelike hypersurfaces, one in the past infinity and the other at an arbitrary time. In cosmological scenarios the Gaussian wave functions on each hypersurface carry the same information as the Heisenberg field operator. In fact, the in-out formalism and the invariant-operator picture formalism are equivalent to each other.

In the Hamiltonian formalism a massive scalar field in the dS space is equivalent to an infinite sum of time-dependent harmonic oscillators.¹⁵ The invariant-operator picture provides a pair of time-dependent annihilation and creation operators that generate all the exact quantum states in analogy with a harmonic oscillator.^{16,17,18,19,20,21} Using these operators we construct the Gaussian wave packets for each oscillator that evolve from the spacelike hypersurface in the past infinity toward to the future infinity.^{17,18,19,20,21} Furthermore, we find the Bogoliubov transformation between the operators on the two different hypersurfaces.

Interestingly, in the planar coordinates of dS space the number of in-vacuum particles carried by the Gaussian wave packet on each hypersurface exponentially increases in proportional to the cube of the scale factor and the vacuum persistence amplitude confirms this phenomenon. This contrasts with the Gibbons-Hawking radiation which is a thermal and scale-factor free distribution in the future infinity. It is shown that the matrix amplitude between the Gaussian wave packet and the out-vacuum Gaussian wave packet in the future infinity, however, exhibits the Gibbons-Hawking radiation. The Gibbons-Hawking radiation is the particle number measured by any detector whose quantum states are prescribed by the out-vacuum solution. Hence, from the view of the in-out formalism this discrepancy raises a fundamental question on the quantum field in the planar coordinates of the dS space.

2. Hamiltonian Formalism for Massive Scalar Field

In a Friedmann-Robertson-Walker spacetime [in the units of $c = \hbar = 1$]

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2, \quad (1)$$

a massive scalar field Φ has the action

$$S = \int dt L = \int dt \int d^3\mathbf{x} \sqrt{-g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - m^2 \Phi^2). \quad (2)$$

In the Hamiltonian formalism the action with $\Pi = a^3 \dot{\Phi}$

$$S = \int dt (\dot{\Phi} \Pi - H), \quad (3)$$

is given by the Hamiltonian and its density

$$H(t) = \int d^3\mathbf{x} \frac{1}{2} \left(\frac{\Pi^2}{a^3} + \frac{(\nabla \Phi)^2}{a^2} + m^2 \Phi^2 \right) = \int d^3\mathbf{x} \mathcal{H}(t, \mathbf{x}). \quad (4)$$

The stress-energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\rho} \left(g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi - m^2 \Phi^2 \right), \quad (5)$$

has also the Hamiltonian expression

$$\begin{aligned} T_{00} &= \frac{\mathcal{H}}{a^3}, \quad T_{0i} = \frac{\Pi}{a^3} \partial_i \Phi, \\ T_{ij} &= \frac{a^2}{2} \delta_{ij} \left(\frac{\Pi^2}{a^6} - m^2 \Phi^2 \right) + \left(\partial_i \Phi \partial_j \Phi - \frac{1}{2} \delta_{ij} (\nabla \Phi)^2 \right). \end{aligned} \quad (6)$$

Now we decompose the Hamiltonian (4) by the Fourier expansion of the field and the conjugate momentum

$$\Phi(t, \mathbf{x}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \Pi(t, \mathbf{x}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \pi_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (7)$$

Then the Hamiltonian is an infinite sum of time-dependent oscillators

$$H(t) = \int d^3 \mathbf{k} \frac{1}{2} \left(\frac{\dot{\pi}_{\mathbf{k}}^2}{a^3} + a^3 \omega_k^2 \phi_{\mathbf{k}}^2 \right) = \int d^3 \mathbf{k} H_k(t), \quad (8)$$

where

$$\pi_k = a^3 \dot{\phi}_k, \quad \omega_k^2(t) = \frac{k^2}{a^2} + m^2. \quad (9)$$

Here and hereafter ϕ_k^2 and π_k^2 denote

$$\begin{aligned} \phi_k^2 &:= \phi_{\mathbf{k}} \phi_{-\mathbf{k}} = \left(\frac{\phi_{\mathbf{k}} + \phi_{-\mathbf{k}}}{2} \right)^2 + \left(\frac{\phi_{\mathbf{k}} - \phi_{-\mathbf{k}}}{2i} \right)^2, \\ \pi_k^2 &:= \pi_{\mathbf{k}} \pi_{-\mathbf{k}} = \left(\frac{\pi_{\mathbf{k}} + \pi_{-\mathbf{k}}}{2} \right)^2 + \left(\frac{\pi_{\mathbf{k}} - \pi_{-\mathbf{k}}}{2i} \right)^2, \end{aligned} \quad (10)$$

and $k^2 = \mathbf{k}^2$. Thus, the k -mode denotes both the cosine and the sine modes. The Hamiltonian (8) has the meaning of $H_k(t) = H_k(\pi_k, \phi_k, \Sigma_t)$ on each spacelike hypersurface Σ_t and the Hamiltonian formalism describes the evolution from one hypersurface Σ_{t_0} to another Σ_t . Similarly, the stress-energy-momentum tensor (6) can be decomposed into the Fourier modes according to eq. (10). The Hamilton equation for $H_k(t)$ is the Fourier mode of the field equation

$$\ddot{\phi}_k + 3 \frac{\dot{a}}{a} \dot{\phi}_k + \omega_k^2(t) \phi_k = 0. \quad (11)$$

3. de Sitter Radiation of Massive Particles in Planar Coordinates

The maximal symmetry of a dS space allows one to explicitly solve the field equation for a massive or massless scalar field. In fact, in the planar coordinates, $a(t) = e^{Ht}$,

$$ds^2 = -dt^2 + e^{2Ht} d\mathbf{x}^2, \quad (12)$$

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eq. (11) in the canonical form takes the form of the Morse potential while in the global coordinates, $a(t) = \cosh(Ht)/H$, its is related to the Pösch-Teller potential. The general solution to eq. (11) in the metric (12) is given by

$$\varphi_k(t) = \left(\frac{\pi}{4H}\right)^{1/2} e^{-3Ht/2} \left[c_k^{(1)} e^{-\pi p/2} H_{ip}^{(1)}(z) + c_k^{(2)} e^{\pi p} H_{ip}^{(2)}(z) \right], \quad (13)$$

where $H_{ip}^{(1)}$ and $H_{ip}^{(2)}$ are the Hankel functions²² and for massive particles ($m/H > 3/2$)

$$z = \frac{k}{H} e^{-Ht}, \quad p = \left(\frac{m^2}{H^2} - \frac{9}{4} \right)^{1/2}. \quad (14)$$

The standard quantization rule requires that the Wronskian condition should be satisfied⁸

$$a^3(t) \text{Wr}[\varphi_k(t), \varphi_k^*(t)] = a^3(t) \left(\varphi_k(t) \dot{\varphi}_k^*(t) - \dot{\varphi}_k(t) \varphi_k^*(t) \right) = i. \quad (15)$$

Though the integration constants $c_k^{(1)}$ and $c_k^{(2)}$ can be arbitrary complex numbers, the quantization rule (15) restricts them to satisfy

$$|c_k^{(1)}|^2 - |c_k^{(2)}|^2 = 1. \quad (16)$$

In the past infinity ($t = -\infty$), the general solution (13) has the asymptotic form

$$\varphi_k(t) = c_k^{(1)} \varphi_k^{(\text{in})}(t) + c_k^{(2)} \varphi_k^{(\text{in})*}(t), \quad (17)$$

where the in-vacuum (the Bunch-Davies vacuum² with $c_k^{(1)} = 1$ and $c_k^{(2)} = 0$) is provided by the positive frequency solution with respect to the energy operator, $i\partial_t$,

$$\varphi_k^{(\text{in})}(t) = \frac{e^{-Ht}}{\sqrt{2k}} \exp\left(i \frac{k}{H} e^{-Ht}\right). \quad (18)$$

In the future infinity ($t = \infty$), using the asymptotic formulae for the Hankel functions²²

$$\begin{aligned} H_{ip}^{(1)}(z) &= J_{ip}(z) + iN_{ip}(z) \\ &= \frac{1}{\sinh(\pi p)} \left[\frac{e^{\pi p}}{\Gamma(1+ip)} \left(\frac{z}{2}\right)^{ip} - \frac{1}{\Gamma(1-ip)} \left(\frac{z}{2}\right)^{-ip} \right], \\ H_{ip}^{(2)}(z) &= J_{ip}(z) - iN_{ip}(z) \\ &= -\frac{1}{\sinh(\pi p)} \left[\frac{e^{-\pi p}}{\Gamma(1+ip)} \left(\frac{z}{2}\right)^{ip} + \frac{1}{\Gamma(1-ip)} \left(\frac{z}{2}\right)^{-ip} \right], \end{aligned} \quad (19)$$

the solution (13) asymptotically becomes

$$\varphi_k(t) = \alpha_k \varphi_k^{(\text{out})}(t) + \beta_k \varphi_k^{(\text{out})*}(t). \quad (20)$$

where the out-vacuum is constructed by another positive frequency solution

$$\varphi_k^{(\text{out})}(t) = \frac{e^{-3Ht/2}}{\sqrt{2Hp}} e^{-iHpt}. \quad (21)$$

Here the Bogoliubov coefficients are given by

$$\begin{aligned}\alpha_k &= \left(\frac{\pi p}{2}\right)^{1/2} \left(c_k^{(1)} e^{\pi p/2} + i c_k^{(1)} e^{-\pi p/2}\right) \frac{(k/2H)^{ip}}{\sinh(\pi p) \Gamma(1+ip)}, \\ \beta_k &= -\left(\frac{\pi p}{2}\right)^{1/2} \left(c_k^{(1)} e^{-\pi p/2} + i c_k^{(1)} e^{\pi p/2}\right) \frac{(k/2H)^{-ip}}{\sinh(\pi p) \Gamma(1-ip)}.\end{aligned}\quad (22)$$

A direct calculation shows that the Bogoliubov relation holds

$$|\alpha_k|^2 - |\beta_k|^2 = \left(|c_k^{(1)}|^2 - |c_k^{(2)}|^2\right) = 1. \quad (23)$$

Noting that $|\beta_k|^2$ is the particle number, the probability for particle production is

$$P_k = \frac{1}{e^{2\pi p} - 1} |c_k^{(1)}|^2 + \frac{e^{2\pi p}}{e^{2\pi p} - 1} |c_k^{(2)}|^2. \quad (24)$$

In the above we have assumed the real $c_k^{(1)}$ and $c_k^{(2)}$ for simplicity which corresponds to the zero squeezing angle. The interpretation of eq. (24) is that the first term is the spontaneous production while the second term is the stimulated emission in the presence of in-vacuum particles. In the specific case of the Bunch-Davies vacuum, the probability for particle production is the Gibbons-Hawking radiation of bosons³

$$P_k = \frac{1}{e^{2\pi p} - 1}. \quad (25)$$

The particle production is independent of the momentum and the scale factor $a(t)$ and is determined by the mass and the Hubble constant.

4. Gaussian Wave Packets

We shall use the Schrödinger picture since quantum states are c-number wave functions that carry the same information as the Heisenberg field operator. Though not considered in this paper, the Schrödinger picture is also convenient for a nonlinear theory of self-interactions, modulo the renormalization of the wave function, the vacuum energy and the coupling constants.

The time-dependent Hamiltonian (8) in an expanding universe may give an adverse feeling for the Schrödinger picture

$$i \frac{\partial}{\partial t} \Psi_k(\phi_k, t) = \hat{H}_k(t) \Psi_k(\phi_k, t). \quad (26)$$

However, it has been known for a long time that the Gaussian wave function could be expressed in terms of a classical solution to eq. (11), for instance, see ref. [23]. It may be understood from the fact that the Heisenberg field operator also satisfies the same equation (11) and the Heisenberg picture provides the same quantum information as the Schrödinger picture. In fact, the Gaussian wave function on a spacelike hypersurface Σ_t of constant t for each Fourier mode, which is the solution to the Schrödinger (26) and is normalized to unity, is given by¹⁸

$$\Psi_k(\phi_k, t) = \left(\frac{\varphi_k}{\sqrt{2\pi}|\varphi_k|^2}\right)^{1/2} \exp\left(\frac{ia^3}{2} \frac{\dot{\varphi}_k^*}{\varphi_k^*} \phi_k^2\right), \quad (27)$$

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where φ_k is a complex solution to eq. (11) that satisfies the quantization rule (15).

In the invariant-operator picture by Lewis and Riesenfeld, the Hilbert space for a time-dependent Hamiltonian of quadratic order can be constructed exactly in the same manner as for a time-independent Hamiltonian. In this picture we look for the operators that satisfy the Liouville-von Neumann equation

$$i\frac{\partial}{\partial t}\hat{b}_k(t) + [\hat{b}_k(t)\hat{H}_k(t)] = 0. \quad (28)$$

Note that eq. (28) is the equation for the density operator and describes the backward evolution

$$\hat{b}_k^{(I)}(t) = \hat{U}_k^\dagger(t)\hat{b}_k^{(S)}\hat{U}_k(t), \quad (29)$$

in contrary to the forward evolution of the Heisenberg operator

$$\hat{b}_k^{(H)}(t) = \hat{U}_k(t)\hat{b}_k^{(S)}\hat{U}_k^\dagger(t), \quad (30)$$

where \hat{U}_k is the evolution operator

$$i\frac{\partial}{\partial t}\hat{U}_k(t) = \hat{H}_k(t)\hat{U}_k(t). \quad (31)$$

Indeed the operator (29) satisfies eq. (28) with respect to the Hamiltonian

$$\hat{H}_k^{(I)}(t) = \hat{U}_k^\dagger(t)\hat{H}_k^{(S)}\hat{U}_k(t). \quad (32)$$

For the quadratic Hamiltonian we can directly find the invariant operators (the superscript (I) dropped below) for eq. (28), which are first order in the momentum and the position operators,^{16,17,18,19,20,21}

$$\begin{aligned} \hat{b}_k(t) &= i(\varphi_k^*(t)\hat{\pi}_k - a^3(t)\dot{\varphi}_k^*(t)\hat{\phi}_k), \\ \hat{b}_k^\dagger(t) &= -i(\varphi_k(t)\hat{\pi}_k - a^3(t)\dot{\varphi}_k(t)\hat{\phi}_k). \end{aligned} \quad (33)$$

In the above $\hat{\pi}_k$ and $\hat{\phi}_k$ are the Schrödinger operators and $\varphi_k(t)$ is a complex solution to eq. (11) together with the quantization rule (15). Remarkably, these operators satisfy the equal-time commutation relation

$$[\hat{b}_k(t), \hat{b}_k^\dagger(t)] = 1, \quad (34)$$

and, moreover, play the role of the time-dependent annihilation and creation operators since they reduce to the standard ones for a time-independent Hamiltonian, modulo time-dependent phase factors. In fact, the Gaussian wave function (27) is annihilated by $\hat{b}_k(t)$.

5. Quantum Dynamics

The Hamiltonian (8) has the algebra SU(1,1), for which we may choose a time-independent Hermitian basis

$$\hat{L}^{(-)} = \frac{1}{2}\hat{\phi}_k^2, \quad \hat{L}^{(+)} = \frac{1}{2}\hat{\pi}_k^2, \quad \hat{L}^{(0)} = \frac{1}{2}(\hat{\phi}_k\hat{\pi}_k + \hat{\pi}_k\hat{\phi}_k). \quad (35)$$

As the field and the conjugate momentum operators have the expression

$$\hat{\phi}_k(t) = \varphi_k(t)\hat{b}_k(t) + \varphi_k^*(t)\hat{b}_k^\dagger(t), \quad \hat{\pi}_k(t) = a^3(t)[\dot{\varphi}_k(t)\hat{b}_k(t) + \dot{\varphi}_k^*(t)\hat{b}_k^\dagger(t)], \quad (36)$$

we may use another Hermitian basis²⁴

$$\hat{M}_k^{(0)}(t) = \hat{b}_k^\dagger(t)\hat{b}_k(t) + \hat{b}_k(t)\hat{b}_k^\dagger(t), \quad (37)$$

$$\hat{M}_k^{(+)}(t) = \hat{b}_k^2(t) + \hat{b}_k^{\dagger 2}(t), \quad \hat{M}_k^{(-)}(t) = i[\hat{b}_k^2(t) - \hat{b}_k^{\dagger 2}(t)]. \quad (38)$$

This basis has the SU(1,1) group structure

$$[\hat{M}_k^{(0)}(t), \hat{M}_{k'}^{(\pm)}(t)] = \pm 2i\hat{M}_k^{(\mp)}(t)\delta_{kk'}, \quad [\hat{M}_k^{(+)}(t), \hat{M}_{k'}^{(-)}(t)] = -2i\hat{M}_k^{(0)}(t)\delta_{kk'}. \quad (39)$$

Note that $\hat{M}_k^{(0)}(t)$ is the number operator on the hypersurface Σ_t .

Using eq. (36) we express the quadratic operators for the field and the conjugate momentum in terms of the basis (37, 38)

$$\begin{aligned} \hat{\phi}_k^2 &= |\varphi_k(t)|^2 \hat{M}_k^{(0)}(t) + \frac{1}{2}(\varphi_k^{*2}(t) + \varphi_k^2(t))\hat{M}_k^{(+)}(t) \\ &\quad + \frac{i}{2}(\varphi_k^{*2}(t) - \varphi_k^2(t))\hat{M}_k^{(-)}(t), \\ \hat{\pi}_k^2 &= a^6(t)[|\dot{\varphi}_k(t)|^2 \hat{M}_k^{(0)}(t) + \frac{1}{2}(\dot{\varphi}_k^{*2}(t) + \dot{\varphi}_k^2(t))\hat{M}_k^{(+)}(t) \\ &\quad + \frac{i}{2}(\dot{\varphi}_k^{*2}(t) - \dot{\varphi}_k^2(t))\hat{M}_k^{(-)}(t)], \\ \hat{\phi}_k\hat{\pi}_k + \hat{\pi}_k\hat{\phi}_k &= a^3(t)[(|\varphi_k(t)|^2)^\cdot \hat{M}_k^{(0)}(t) + \frac{1}{2}((\varphi_k^{*2}(t))^\cdot + (\varphi_k^2(t))^\cdot)\hat{M}_k^{(+)}(t) \\ &\quad + \frac{i}{2}((\varphi_k^{*2}(t))^\cdot - (\varphi_k^2(t))^\cdot)\hat{M}_k^{(-)}(t)]. \end{aligned} \quad (40)$$

Conversely, the number operator can be expressed in the basis (35) as

$$\frac{1}{2}\hat{M}_k^{(0)}(t) = |\varphi_k(t)|^2 \hat{\pi}_k^2 + a^6(t)|\dot{\varphi}_k(t)|^2 \hat{\phi}_k^2 - a^3(t)(|\varphi_k(t)|^2)^\cdot \frac{1}{2}(\hat{\phi}_k\hat{\pi}_k + \hat{\pi}_k\hat{\phi}_k). \quad (41)$$

The quadratic variances with respect to the Gaussian wave function (27) on Σ_t take the values

$$\begin{aligned} \langle \Psi_k(t) | \hat{\phi}_k^2 | \Psi_k(t) \rangle &= |\varphi_k(t)|^2, \quad \langle \Psi_k(t) | \hat{\pi}_k^2 | \Psi_k(t) \rangle = a^6(t)|\dot{\varphi}_k(t)|^2, \\ \langle \Psi_k(t) | \hat{\phi}_k\hat{\pi}_k + \hat{\pi}_k\hat{\phi}_k | \Psi_k(t) \rangle &= a^3(t)(|\varphi_k(t)|^2)^\cdot. \end{aligned} \quad (42)$$

6. Particle Production in Quantum Dynamics

We may raise the question whether the operator (37) indeed provides a proper definition for the particle number

$$\hat{N}_k^\dagger(t) = \hat{b}_k^\dagger(t)\hat{b}_k(t) = \frac{1}{2}\hat{M}_k^{(0)}(t) - \frac{1}{2}. \quad (43)$$

The Gaussian wave function (27) has the zero particle number on Σ_t

$$\langle \Psi_k(t) | \hat{N}_k(t) | \Psi_k(t) \rangle = 0. \quad (44)$$

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Thus the number operator $\hat{N}_k^\dagger(t)$ measures the amount of particles carried by the Gaussian wave function on each Σ_t through the evolution, which is a conserved quantum number, that is, the harmonic excitation number. We use the Gaussian wave function on the hypersurface $\Sigma_{-\infty}$ in the past infinity as the in-vacuum state for each k and define the in-vacuum as the tensor product

$$|\text{vac, in}\rangle = \prod_k |\Psi_k(-\infty)\rangle. \quad (45)$$

Similarly we define the out-vacuum as

$$|\text{vac, out}\rangle = \prod_k |\Psi_k(\infty)\rangle. \quad (46)$$

Then the number of in-vacuum particles contained in the Gaussian wave function on Σ_t or the number of out-vacuum particles contained in the in-vacuum Gaussian wave function on $\Sigma_{-\infty}$ is

$$n_k(t) = \langle \Psi_k(t) | \hat{N}_k(-\infty) | \Psi_k(t) \rangle = \langle \Psi_k(-\infty) | \hat{N}_k(t) | \Psi_k(-\infty) \rangle. \quad (47)$$

The second equality is a consequence of the reciprocity, as will be explicitly shown below.

Taking the expectation value of the operator $\hat{M}_k^{(0)}(-\infty)$ in eq. (41) with respect to the Gaussian wave function on Σ_t , we obtain

$$\begin{aligned} n_k(t) &= a^6(t) |\varphi_k(-\infty)|^2 |\dot{\varphi}_k(t)|^2 + a^6(-\infty) |\dot{\varphi}_k(-\infty)|^2 |\varphi_k(t)|^2 \\ &\quad - \frac{1}{2} a^3(-\infty) a^3(t) (|\varphi_k(-\infty)|^2) \cdot (|\varphi_k(t)|^2) - \frac{1}{2}. \end{aligned} \quad (48)$$

We find the Bogoliubov transformation between Σ_{t_0} and Σ_t

$$\begin{aligned} \hat{a}_k(t_0) &= \mu_k(t_0, t) \hat{a}_k(t) + \nu_k(t_0, t) \hat{a}_k^\dagger(t), \\ \hat{a}_k^\dagger(t_0) &= \mu_k^*(t_0, t) \hat{a}_k^\dagger(t) + \nu_k^*(t_0, t) \hat{a}_k(t), \end{aligned} \quad (49)$$

where

$$\begin{aligned} \mu_k(t_0, t) &= i[a^3(t) \varphi_k^*(t_0) \dot{\varphi}_k(t) - a^3(t_0) \varphi_k(t) \dot{\varphi}_k^*(t_0)], \\ \nu_k(t_0, t) &= i[a^3(t) \varphi_k^*(t_0) \dot{\varphi}_k^*(t) - a^3(t_0) \varphi_k^*(t) \dot{\varphi}_k(t_0)]. \end{aligned} \quad (50)$$

The Bogoliubov transformation (49) is symmetric under the interchange of $\hat{a}_k(t_0)$, $\hat{a}_k^\dagger(t_0)$ and $\hat{a}_k(t)$, $\hat{a}_k^\dagger(t)$, so the inverse transformation

$$\begin{aligned} \hat{a}_k(t) &= \mu_k(t, t_0) \hat{a}_k(t_0) + \nu_k(t, t_0) \hat{a}_k^\dagger(t_0), \\ \hat{a}_k^\dagger(t) &= \mu_k^*(t, t_0) \hat{a}_k^\dagger(t_0) + \nu_k^*(t, t_0) \hat{a}_k(t_0), \end{aligned} \quad (51)$$

has the coefficients

$$\mu_k(t, t_0) = \mu_k^*(t_0, t), \quad \nu_k(t, t_0) = -\nu_k(t_0, t). \quad (52)$$

It can be shown that the particle number is given by

$$n_k(t) = |\nu_k(-\infty, t)|^2 = |\nu_k(t, -\infty)|^2. \quad (53)$$

Furthermore, the Bogoliubov relation holds

$$|\mu_k(t_0, t)|^2 - |\nu_k(t_0, t)|^2 = 1. \quad (54)$$

However, the Bogoliubov coefficients (22) cannot be obtained from those (50) using the asymptotic solutions in section 3.

7. Vacuum Persistence Amplitude

One important concept for the particle production and the vacuum instability is the vacuum persistence amplitude introduced by Schwinger and DeWitt,^{7,8} which is the scattering-matrix (S -matrix) amplitude between two different hypersurfaces

$$e^{iW_k(t, t_0)} = \langle \Psi_k(t) | \Psi_k(t_0) \rangle. \quad (55)$$

In particle physics the S -matrix takes two asymptotic limits $t_0 = -\infty$ and $t = \infty$, that is, before and after interactions. But it is still legitimate to use the definition (55) in an intermediate region, though the physical meaning requires further clarification and investigation. The integration of two Gaussian wave functions (27) gives

$$W_k(t, t_0) = i \ln(\mu_k(t_0, t)) = i \ln(\mu_k^*(t, t_0)), \quad (56)$$

where μ_k is given in eqs. (50, 52) and the mode k counts both \mathbf{k} and $-\mathbf{k}$.

The vacuum persistence amplitude for the field is the product of eq. (55)

$$e^{iW(t, t_0)} = \prod_k \langle \Psi_k(t) | \Psi_k(t_0) \rangle, \quad (57)$$

and thereby the effective action is

$$W(t, t_0) = i\mathcal{V}(t) \int d^3\mathbf{k} \ln(\mu_{\mathbf{k}}(t_0, t)) = i\mathcal{V}(t) \int d^3\mathbf{k} \ln(\mu_{\mathbf{k}}^*(t, t_0)). \quad (58)$$

Here $\mu_{\mathbf{k}}(t_0, t)$ is the Bogoliubov coefficient obtained by substituting \mathbf{k} for k in eq. (50) and $\mathcal{V}(t)$ is a spatial volume relevant to the hypersurface on Σ_t and may be chosen as $\mathcal{V}(t) = a^3(t) \int d^3\mathbf{x}$. The effective action (58) involves the divergent terms that renormalize the vacuum energy or the mass, and the gravitational constant or the charge. In the in-out formalism equation eq. (58) has been used to find the effective action for the massive scalar field in the global geometry of dS space¹⁴ and in quantum electrodynamics (QED) the effective action in a time-dependent electric field.²⁵

The vacuum persistence (twice of the imaginary part) relates to the particle number through the relation

$$2 \text{Im}(W_k(t, t_0)) = \ln(|\mu_k(t, t_0)|^2) = \ln(1 + |\nu_k(t, t_0)|^2). \quad (59)$$

And the probability for the in-state on Σ_{t_0} to remain in the out-state on Σ_t is

$$|\langle 0, t | 0, t_0 \rangle|^2 = e^{-2\text{Im}(W(t, t_0))} = \exp\left[-\mathcal{V}(t) \int d^3\mathbf{k} \ln(1 + n_{\mathbf{k}}(t, t_0)^2)\right]. \quad (60)$$

Thus the vacuum of dS space becomes unstable due to the dS radiation. It is shown that the number of in-vacuum particles carried by the Gaussian wave function on at a later time hypersurface Σ_t is dominated by the first term in eq. (48)

$$n_k(t) \simeq a^3(t) (a^3(t) |\dot{\varphi}_k(t)|^2) |\dot{\varphi}_k(t)|^2 |\varphi_k(-\infty)|^2 \quad (61)$$

and is enhanced by the exponential scale factor and by the infinitely blueshifted wavelength on the initial surface while the parenthesis is finite. This catastrophic production of in-vacuum particles in the infinity future is a generic phenomenon in an expanding universe with a cosmological singularity. An interesting observation is that if a regularization scheme removes $a^3(t)$ and $|\varphi_k(-\infty)|^2$ in the vacuum persistence probability (60), the remaining term is related to the Gibbons-Hawking radiation.

8. Wave Packet for Out-Vacuum

The out-vacuum solution,

$$\varphi_k^{(\text{out})}(t) = \left(\frac{\pi}{2H \sinh(\pi p)} \right)^{1/2} e^{-3Ht/2} J_{ip}(z), \quad (62)$$

satisfies the quantization rule (15) and has the same time-dependent factor as eq. (20) up to a constant phase factor. The corresponding Gaussian wave function

$$\Psi_k^{(\text{out})}(\phi_k, t) = \left(\frac{\varphi_k^{(\text{out})}}{\sqrt{2\pi} |\varphi_k^{(\text{out})}|^2} \right)^{1/2} \exp\left(\frac{ia^3}{2} \frac{\dot{\varphi}_k^{(\text{out})*}}{\varphi_k^{(\text{out})*}} \phi_k^2 \right), \quad (63)$$

is another exact solution to the Schrödinger equation (26) since eq. (62) is a complex solution to eq. (11). The Bunch-Davies vacuum solution (13) ($c_k^{(1)} = 1$, $c_k^{(2)} = 0$) now has the Bogoliubov transformation

$$\varphi_k(t) = \alpha_k \varphi_k^{(\text{out})}(t) + \beta_k \varphi_k^{(\text{out})*}(t), \quad (64)$$

where

$$\alpha_k = \left(\frac{e^{\pi p}}{2 \sinh(\pi p)} \right)^{1/2}, \quad \beta_k = - \left(\frac{e^{-\pi p}}{2 \sinh(\pi p)} \right)^{1/2}. \quad (65)$$

The particle production $|\beta_k|^2$ is the Gibbons-Hawking radiation for massive bosons (25).

In analogy with the vacuum persistence amplitude between the in-vacuum and the out-vacuum in section 7, we may compute the matrix amplitude on the same hypersurface Σ_t

$$e^{iW_k^{(\text{out})}(t)} = \langle \Psi_k(t) | \Psi_k^{(\text{out})}(t) \rangle. \quad (66)$$

Then the vacuum persistence (twice of the imaginary part) relates to the particle number through the relation

$$2 \text{Im}(W_k^{(\text{out})}(t)) = \ln(|\alpha_k|^2) = \ln(1 + |\beta_k|^2), \quad (67)$$

and explains the Gibbons-Hawking radiation.

9. Conclusion

In this paper we have studied the production of massive particles in a dS space using the quantum dynamics that unifies the invariant-operator picture with the in-out formalism. In the quantum dynamics a massive field in an expanding FRW universe or a dS space is equivalent to an infinite sum of time-dependent oscillators in which the time-dependent mass comes from the expanding spatial volume and the frequencies from the redshifted or blueshifted wavelengths. The main advantage of the invariant-operator picture is that there exist the time-dependent annihilation and creation operators which generate the exact quantum states, the Gaussian wave function being the simplest one among them. The Gaussian wave function provides the in-vacuum in the past infinity and the out-vacuum in the future infinity. Hence the time-dependent annihilation and creation operators connect these two asymptotic regions through the Bogoliubov transformation.

We have found the Gaussian wave packets corresponding to the Bunch-Davies vacuum and its one-parameter family. The scattering-matrix of Gaussian wave packets between two different spacelike hypersurfaces directly gives the vacuum persistence amplitude, which is equivalent to that from the Bogoliubov transformation and its coefficients. The vacuum persistence amplitude is an important tool in quantifying how the massive field probes the background dS space in analogy with QED in a strong electric field background. However, in the planar coordinates of dS space the number of in-vacuum particles carried by the Gaussian wave packet exponentially increases in proportion to the expanding spatial volume, in contrast with the Gibbons-Hawking radiation. This is because the momentum variance, the leading term for particle number, exponentially increases in the infinity future while the wave packet is very sharply peaked in the field. On the other hand, the matrix amplitude between the Gaussian wave packet and the out-vacuum Gaussian wave packet on the same hypersurface in the far future gives the vacuum persistence that explains the Gibbons-Hawking radiation. This raises a fundamental question which vacuum persistence amplitude is the proper definition for the effective action and the dS radiation. The related issues including the massless scalar field will be addressed in a future publication.

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